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LETTER TO THE EDITOR

Quantum $m \times n$-matrices and q-deformed Binet–Cauchy formula

S A Merkulov

Theoretical Problems Department, USSR Academy of Sciences, ul. Vesnina 12, Moscow 121002, USSR

Abstract. Quantum multiplicative matrices of size $m \times n$ are introduced and studied. The q-generalization of the Binet–Cauchy formula is found.

Recently there has been growing interest [1–5], both from the physical as well as the mathematical point of view, in studying quantum 'groups', 'spaces' and algebras. Much attention has been paid to square quantum multiplicative matrices which are the key objects in the quantum group theory.

In this letter we define and investigate quantum multiplicative matrices of size $m \times n$. In particular, we find the q-generalization of the classical Binet–Cauchy formula, which expresses the quantum determinant of the matrix product $\mathbf{U} \circ \mathbf{V}$ of a quantum matrix $\mathbf{U}$ of size $m \times n$ by a quantum matrix $\mathbf{V}$ of size $n \times m$ in terms of mth order q-minors of $\mathbf{U}$ and $\mathbf{V}$.

Following [1], we first make a brief review of definitions of quantum groups $M_q(n)$ of matrices of size $n \times n$ and the corresponding quantum planes $A_q^{n|0}$ and $A_q^{0|n}$.

The coordinate ring $A_q(n)$ of the manifold $M_q(n)$ of quantum $n \times n$-matrices is the polynomial $\mathbb{C}$-algebra $\mathbb{C}\langle z_i^k,\, i, k = 1, \ldots, n\rangle$ generated by n^2 symbols z_i^k subject to the following relations

$$\begin{aligned} &z_i^k z_i^l = q z_i^l z_i^k && \text{for } k<l \\ &z_i^k z_j^k = q z_j^k z_i^k && \text{for } i<j \\ &z_i^k z_j^l - z_j^l z_i^k = (q-q^{-1}) z_i^l z_j^k && \text{for } i<j,\ k<l \\ &z_i^k z_j^l = z_j^l z_i^k && \text{for } i<j,\ k>l. \end{aligned} \tag{1}$$

The quantum determinant of the matrix z_i^j generating $A_q(n)$ is defined by

$$\begin{aligned} \det_q(z_i^j) &= \sum_{\sigma \in S_n} (-q)^{l(\sigma)} z_1^{\sigma(1)} z_2^{\sigma(2)} \ldots z_n^{\sigma(n)} \\ &= \sum_{\sigma \in S_n} (-q)^{l(\sigma)} z_{\sigma(1)}^1 z_{\sigma(2)}^2 \ldots z_{\sigma(n)}^n \end{aligned} \tag{2}$$

where S_n is the permutation group of the set $\{1, 2, \ldots, n\}$ and, for each $\sigma \in S_n$, $l(\sigma)$ denotes the number of pairs (i, j) with $1 \leqslant i < j \leqslant n$ and $\sigma(i) > \sigma(j)$.

According to Manin [5], the quantum group $M_q(n)$ may be described as an automorphism 'group' of quantum planes $A_q^{n|0}$ and $A_q^{0|n}$ which are polynomial $\mathbb{C}$-algebras, $A_q^{n|0} = \mathbb{C}\langle x^i, i = 1, \ldots, n\rangle$ and $A_q^{0|n} = \mathbb{C}\langle \xi^j, j = 1, \ldots, n\rangle$, generated correspondingly by symbols x^i and ξ^j with the commutation rules

$$x^i x^j = q x^j x^i \qquad \text{for } i<j \tag{3}$$

$$(\xi^i)^2 = 0 \qquad \xi^i \xi^j = -q^{-1} \xi^j \xi^i \qquad \text{for } i<j. \tag{4}$$

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More exactly, there exist algebra morphisms

$$\delta_n : A_q^{n|0} \to A_q(n) \otimes A_q^{n|0}, \ \tilde{\delta}_n : A_q^{0|n} \to A_q(n) \otimes A_q^{0|n}$$

such that

$$\delta_n(x^i) = \sum_{j=1}^{n} z_j^i \otimes x^j \qquad \tilde{\delta}_n(\xi^i) = \sum_{j=1}^{n} z_j^i \otimes \xi^j.$$

The co-action $\tilde{\delta}_n$ applied to the monomial $\xi^1 \ldots \xi^n$ gives the formula [1, 5]

$$\tilde{\delta}_n(\xi_1 \ldots \xi_n) = \mathrm{Det}_q(z_j^i) \otimes \xi^1 \ldots \xi^n.$$

Analogously, applying $\tilde{\delta}_n$ to the monomial $\xi^{i_1} \ldots \xi^{i_n}$ one finds

$$\begin{aligned}\tilde{\delta}_n(\xi^{i_1} \ldots \xi^{i_m}) &= \sum_{j_1, \ldots, j_n} z_{j_1}^{i_1} \ldots z_{j_n}^{i_n} \otimes \xi^{j_1} \ldots \xi^{j_n} \\ &= \sum_{\sigma \in S_n} (-q)^{l(\sigma)} z_{\sigma(1)}^{i_1} \ldots z_{\sigma(n)}^{i_n} \otimes \xi^1 \ldots \xi^n. \end{aligned} \tag{5}$$

If two or more indices $i_1, \ldots, i_n$ coincide, the left-hand side of (5) vanishes. Thus we obtain the formula

$$\sum_{\sigma \in S_n} (-q)^{l(\sigma)} z_{\sigma(1)}^{i_1} \ldots z_{\sigma(n)}^{i_n} = 0 \qquad \text{if } i_k = i_l \text{ for some } k, l \in \{1, \ldots, n\}.$$

If all indices $i_1, \ldots, i_n$ are distinct, we obtain from (5) the identity

$$\sum_{\sigma \in S_n} (-q)^{l(\sigma)} z_{\sigma(1)}^{i_1} \ldots z_{\sigma(n)}^{i_n} = (-q)^{l[i_1 \ldots i_n]} \mathrm{Det}_q(z_i^j) \tag{6}$$

where $l[i_1 \ldots i_n]$ denotes the number of pairs (i_k, i_l) in the ordered set $\{i_1, \ldots, i_n\}$ with $k < l$ and $i_k > i_l$.

Manin's interpretation of $M_q(n)$ suggests to define the space $M_q(m, n)$ of quantum matrices of size $m \times n$ as the 'space' of algebra morphism from quantum planes $A_q^{m|0}$ and $A_q^{0|m}$ to, respectively, quantum planes $A_q^{n|0}$ and $A_q^{0|n}$.

Thus we introduce symbols U_i^α (Latin indices take values $1, \ldots, n$, while Greek ones take values $1, \ldots, m$) and consider the polynomial $\mathbb{C}$-algebra $A_q(m, n) = \mathbb{C} < U_i^\alpha, 1 \leq \alpha \leq m, 1 \leq i \leq n \rangle$. Let us find commutation relations for U_i^α that ensure the existence of algebra morphisms

$$\delta_{m,n} : A_q^{m|0} \to A_q(m, n) \otimes A_q^{n|0} \qquad \tilde{\delta}_{m,n} : A_q^{0|m} \to A_q(m, n) \otimes A_q^{n|0}$$

such that

$$\delta_{m,n}(y^\alpha) = \sum_{i=1}^{n} U_i^\alpha \otimes x^i \qquad \tilde{\delta}_{m,n}(\zeta^\alpha) = \sum_{i=1}^{n} U_i^\alpha \otimes \xi^i$$

where $y^\alpha, \zeta^\beta, x^i$ and ξ^j are canonical generators of $A_q^{m|0}, A_q^{0|m}, A_q^{n|0}$ and $A_q^{0|n}$ respectively.

Simple calculations give the following result: the desired morphisms $\delta_{m,n}$ and $\tilde{\delta}_{m,n}$ exist provided symbols U_i^α satisfy the commutation relations

$$\begin{aligned} &U_i^\alpha U_i^\beta = q U_i^\beta U_i^\alpha && \text{for } \alpha < \beta \\ &U_i^\alpha U_j^\alpha = q U_j^\alpha U_i^\alpha && \text{for } i < j \\ &U_i^\alpha U_j^\beta = U_j^\beta U_i^\alpha && \text{for } \alpha < \beta \text{ and } i > j, \text{ or } \beta > \alpha \text{ and } i < j \\ &U_i^\alpha U_j^\beta - U_j^\beta U_i^\alpha = (q - q^{-1}) U_j^\alpha U_i^\beta && \text{for } \alpha < \beta, i < j. \end{aligned} \tag{7}$$

The set of symbols $\mathbf{U}=\{U_i^\alpha\}$ satisfying the commutation algebra (7) is called a quantum matrix of size $m\times n$, while the correspondent polynomial ring $A_q(m,n)=\mathbb{C}\langle U_i^\alpha\rangle$ is called the coordinate ring on the manifold $M_q(m,n)$ of quantum matrices of size $m\times n$.

Suppose that $\mathbf{U}=\{U_i^\alpha\}\in M_q(m,n)$ and $\mathbf{V}=\{V_A^i\}\in M_q(n,p)$ (capital Latin indices take values $1,\ldots,p\}$, and consider the $m\times p$ matrix $\mathbf{W}=\{W_A^\alpha\}$ given by

$$W_A^\alpha=\sum_{i=1}^{n} U_i^\alpha\otimes V_A^i. \tag{8}$$

Using commutation relations (7) for $\{U_i^\alpha\}$ and $\{V_A^i\}$, one obtains, if $\alpha<\beta$,

$$\begin{aligned} W_A^\alpha W_A^\beta &= \sum_{i,j=1}^{n} U_i^\alpha U_j^\beta\otimes V_A^i V_A^j \\ &= \sum_{i<j} U_i^\alpha U_j^\beta\otimes V_A^i V_A^j+\sum_{i=1}^{n} U_i^\alpha U_i^\beta\otimes V_A^i V_A^i+\sum_{i>j} U_i^\alpha U_j^\beta\otimes V_A^i V_A^j \\ &= q\sum_{i<j} U_j^\beta U_i^\alpha\otimes V_A^j V_A^i+(q-q^{-1})\sum_{i<j} U_i^\beta U_j^\alpha\otimes V_A^i V_A^j \\ &\quad +q\sum_{i=1}^{n} U_i^\alpha U_i^\beta\otimes V_A^i V_A^i+q^{-1}\sum_{i>j} U_j^\beta U_i^\alpha\otimes V_A^j V_A^i=qW_A^\beta W_A^\alpha. \end{aligned}$$

Analogously, one finds

$$\begin{aligned} &W_A^\alpha W_B^\alpha=qW_B^\alpha W_A^\alpha && \text{if } A<B \\ &W_A^\alpha W_B^\beta=W_B^\beta W_A^\alpha && \text{if } \alpha<\beta \text{ and } A>B, \text{ or } \alpha>\beta \text{ and } A<B \\ &W_A^\alpha W_B^\beta-W_B^\beta W_A^\alpha=(q-q^{-1})W_B^\alpha W_A^\beta && \text{if } \alpha<\beta,\ A<B. \end{aligned}$$

Therefore, we conclude that the matrix $\mathbf{W}=\mathbf{U}\circ\mathbf{V}$, where $\circ$ denotes the matrix multiplication (8), is a quantum matrix of size $m\times p$.

This important property of multiplicativity of non-square quantum matrices may be restated in the spirit of Hopf algebras as follows: for any natural numbers m, n and p there exist algebra morphisms

$$\Delta_{m,n,p}: A_q(m,p)\to A_q(m,n)\otimes A_q(n,p)$$

such that

$$\Delta_{m,n,p}(W_A^\alpha)=\sum_{i=1}^{n} U_i^\alpha\otimes V_A^i$$

where $\{W_A^\alpha\}$, $\{U_i^\alpha\}$ and $\{V_A^i\}$ are canonical generators of coordinate rings $A_q(m,p)$, $A_q(m,n)$ and $A_q(n,p)$ respectively.

Moreover, morphisms $\Delta_{m,n,p}$ and defined earlier morphisms $\delta_{m,n}$ and $\delta_{n,p}$ are related to each other in such a way that one has, for any m, n and p, a commutative diagram

$$\begin{array}{ccc} A_q^{m|0} & \xrightarrow{\ \delta_{m,p}\ } & A_q(m,p)\otimes A_q^{p|0} \\ {\scriptstyle\delta_{m,n}}\downarrow & & \downarrow{\scriptstyle\Delta_{m,n,p}\otimes\mathrm{id}} \\ A_q(m,n)\otimes A_q^{n|0} & \xrightarrow{\ \mathrm{id}\otimes\delta_{n,p}\ } & A_q(m,n)\otimes A_q(n,p)\otimes A_q^{p|0}. \end{array}$$

There is also an analogous diagram for the family of morphisms $\tilde{\delta}_{m,n}$.

In the particular case $m=n=p$ the comultiplication $\Delta_{m,n,p}$ coincides precisely with the comultiplication Δ which enters the definition of the quantum group [1-5].

Let $\mathbf{U}=\{U_i^\alpha\}$ be a quantum matrix of size $m\times n$ (with $m<n$) and $\mathbf{V}=\{V_\alpha^i\}$ a quantum matrix of size $n\times m$, so that the quantum matrix $\mathbf{W}=\mathbf{U}\circ\mathbf{V}$ is of size $n\times n$, and we may consider its quantum determinant $\mathrm{Det}_q\mathbf{W}$. Our task now is to express $\mathrm{Det}_q\mathbf{W}$ in terms of constituent quantum matrices $\mathbf{U}$ and $\mathbf{V}$.

Fix any ordered set $1\leq i_1<i_2<\ldots<i_m\leq n$ of m natural numbers $i_1,\ldots,i_m$ and define the correspondent q-minor of the mth order of $\mathbf{U}$ by

$$\mathbf{U}_q(i_1\ldots i_m)=\sum_{\sigma\in S_m}(-q)^{l(\sigma)}U_{i_1}^{\sigma(1)}\ldots U_{i_m}^{\sigma(m)} \tag{9}$$

and the q-minor of the mth order of $\mathbf{V}$ by

$$\mathbf{V}_q(i_1\ldots i_m)=\sum_{\sigma\in S_m}(-q)^{l(\sigma)}V_{\sigma(1)}^{i_1}\ldots V_{\sigma(m)}^{i_m}. \tag{10}$$

By definition (2),

$$\begin{aligned}\mathrm{Det}_q(\mathbf{W})&=\sum_{\sigma\in S_m}(-q)^{l(\sigma)}W_1^{\sigma(1)}\ldots W_n^{\sigma(n)}\\&=\sum_{\sigma\in S_m}(-q)^{l(\sigma)}U_{j_1}^{\sigma(1)}\ldots U_{j_n}^{\sigma(n)}\otimes V_1^{j_1}\ldots V_n^{j_n}.\end{aligned} \tag{11}$$

Terms in the sum (11) which have two or more indices $j_1,\ldots,j_m$ coinciding, equal to zero. Thus we should consider only those $n!/(n-m)!$ summands whose indices $j_1,\ldots,j_n$ are pairwise distinct. Then the sum (11) may be represented as a union of $\binom{n}{m}$ sums, each having $m!$ summands differing from each other only by a permutation of a fixed set of indices $\{j_1,\ldots,j_m\}$ with $j_1<\ldots<j_m$.

Let us consider one of such sums,

$$G(j_1\ldots j_m)\equiv\sum_{\sigma\in S_m}\sum_{\sigma'\in S'_m}(-q)^{l(\sigma)}U_{\sigma'(j_1)}^{\sigma(1)}\ldots U_{\sigma'(j_m)}^{\sigma(m)}\otimes V_1^{\sigma'(j_1)}\ldots V_m^{\sigma'(j_m)}$$

where S'_m is the permutation group of the set $\{j_1,\ldots,j_m\}$. By (6), one has

$$\begin{aligned}G(j_1\ldots j_m)&=\sum_{\sigma\in S'_m}U_q(j_1\ldots j_m)^{l[\sigma'(j_1)\ldots\sigma'(j_m)]}\otimes V_1^{\sigma'(j_1)}\ldots V_m^{\sigma'(j_m)}\\&=U_q(j_1\ldots j_m)\otimes V_q(j_1\ldots j_m).\end{aligned}$$

Therefore we conclude that

$$\mathrm{Det}_q(\mathbf{U}\circ\mathbf{V})=\sum_{j_1<\ldots<j_m}U_q(j_1\ldots j_m)\otimes V_q(j_1\ldots j_m).$$

This is the q-generalization of the classical Binet–Cauchy formula [6]. It can be readily generalized as follows.

Let $\mathbf{U}=\{U_i^\alpha\}$ be a quantum matrix of size $m\times n$ and $\mathbf{V}=\{V_A^i\}$ a quantum matrix of size $m\times p$ so that the quantum matrix $\mathbf{W}=\mathbf{U}\circ\mathbf{V}$ is of size $m\times p$. Define q-minors of the rth order of $\mathbf{U}$ by

$$U_q\begin{pmatrix}i_1\ldots i_r\\ j_1\ldots j_r\end{pmatrix}=\mathrm{Det}_q\{U_{i_t}^{\alpha_s},1\leq s,t\leq m\}$$

and similarly for q-minors of $\mathbf{V}$ and $\mathbf{W}$. Then

$$W\begin{pmatrix}i_1\ldots i_r\\ k_1\ldots k_r\end{pmatrix}=\sum_{j_1<\ldots<j_r}U_q\begin{pmatrix}i_1\ldots i_r\\ j_1\ldots j_r\end{pmatrix}\otimes V_q\begin{pmatrix}j_1\ldots j_r\\ k_1\ldots k_r\end{pmatrix}.$$

In the limit $q\to1$ the latter formula also reduces to the well known classical result [6].

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